Isogeny-based cryptography: A brand new day

Central European Conference on Cryptology

Thomas Decru

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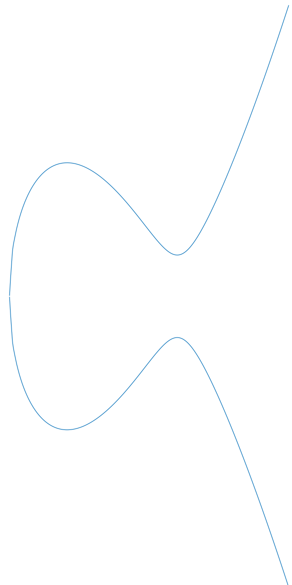
Elliptic curve group law



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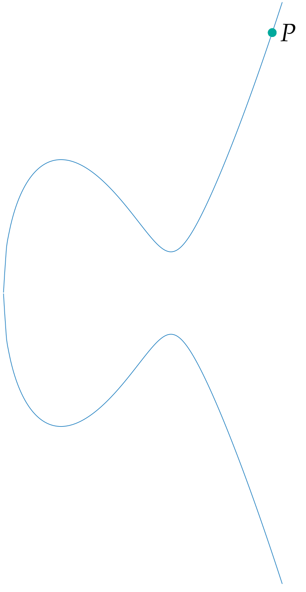

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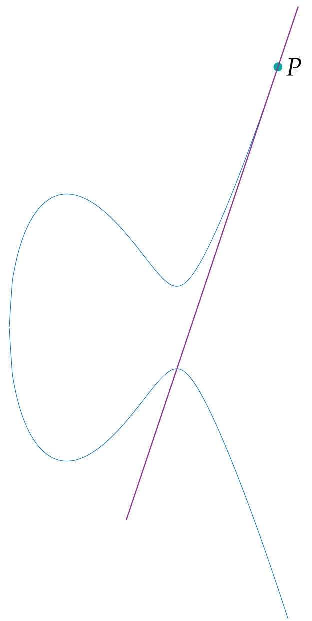

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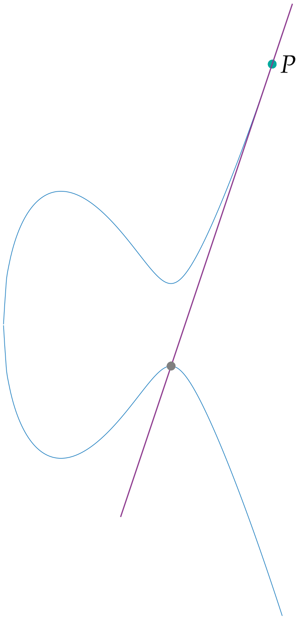

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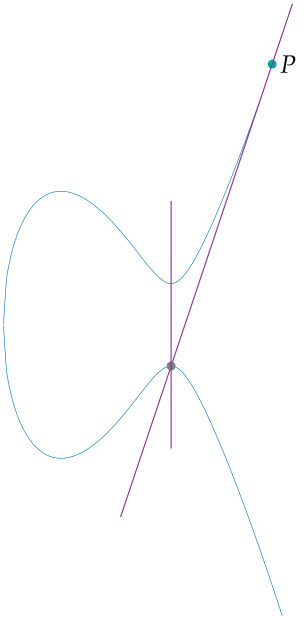

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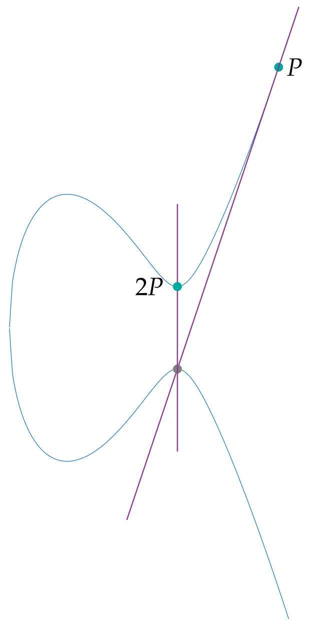

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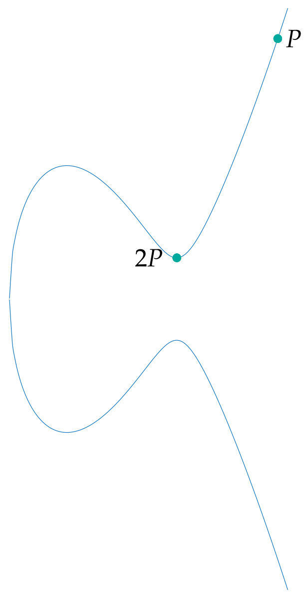

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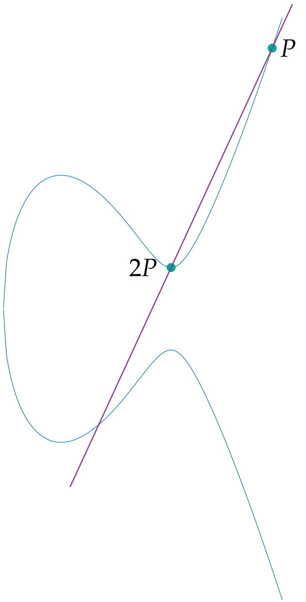

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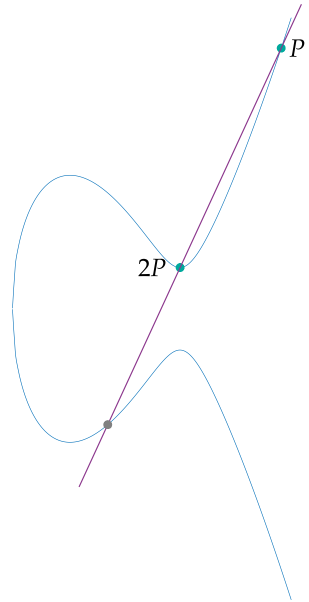

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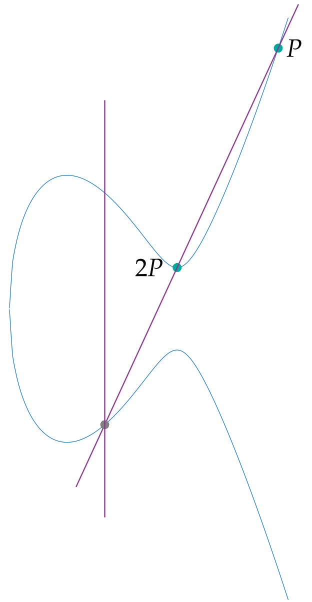

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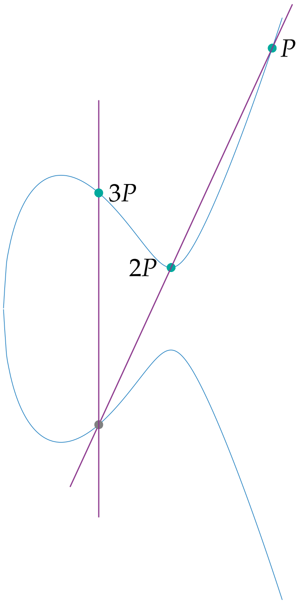

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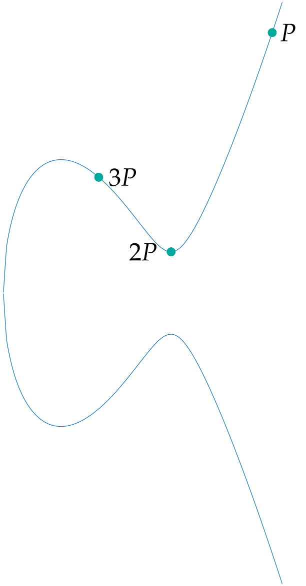

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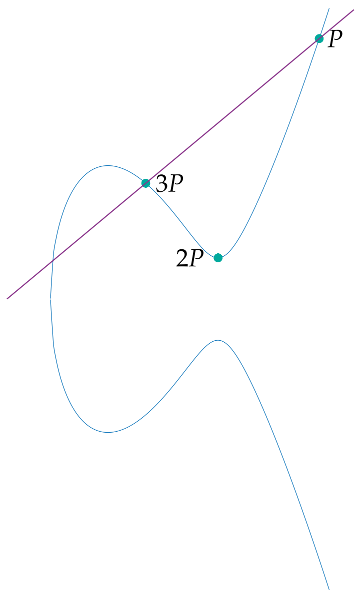

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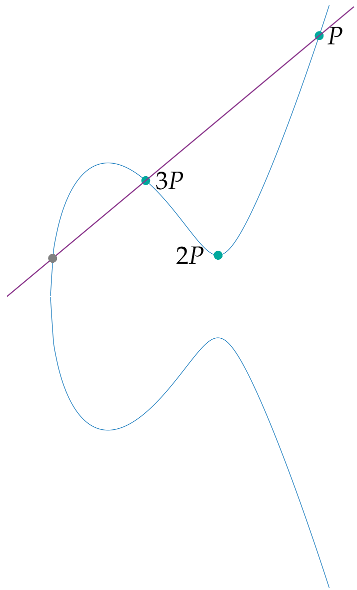

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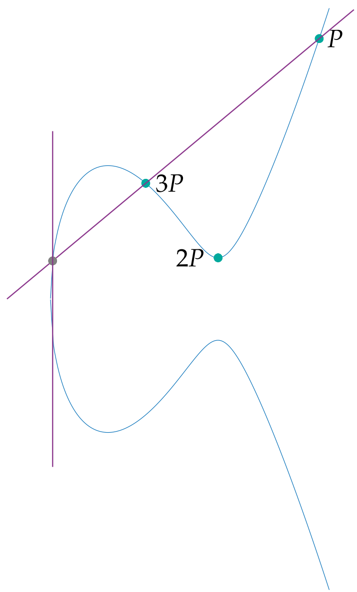

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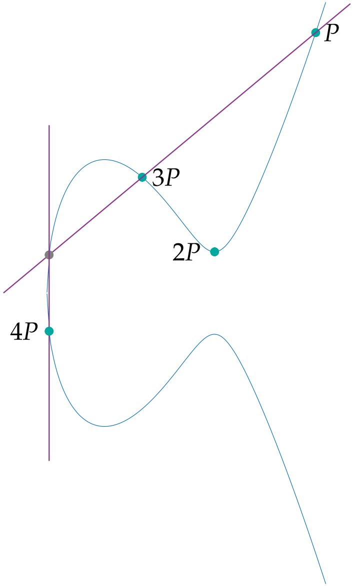

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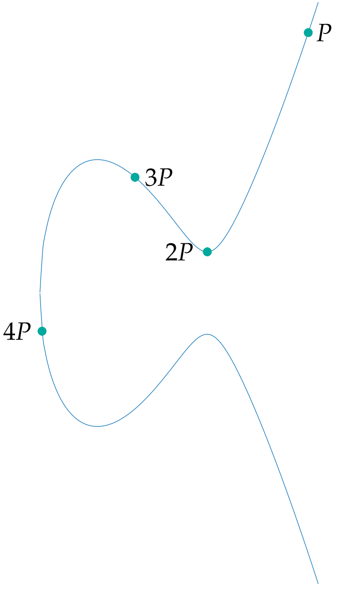

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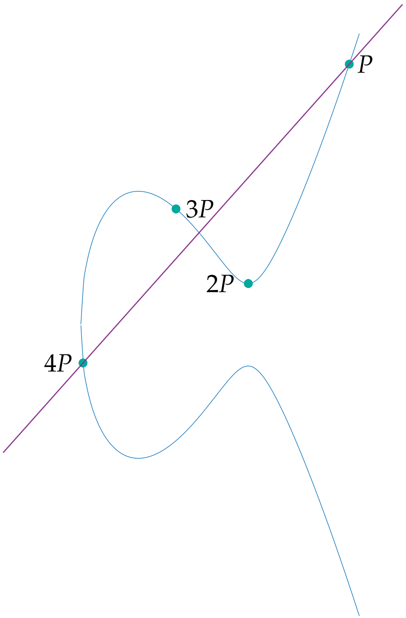

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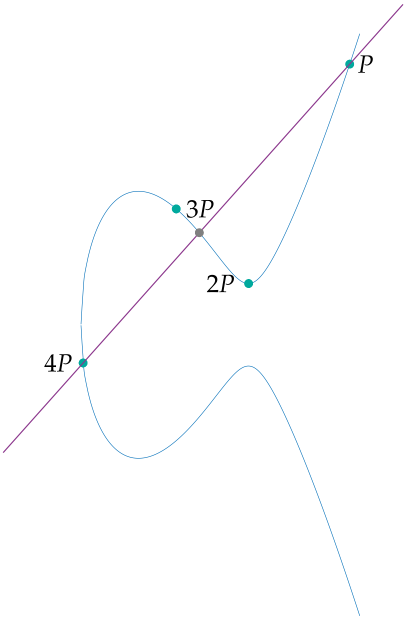

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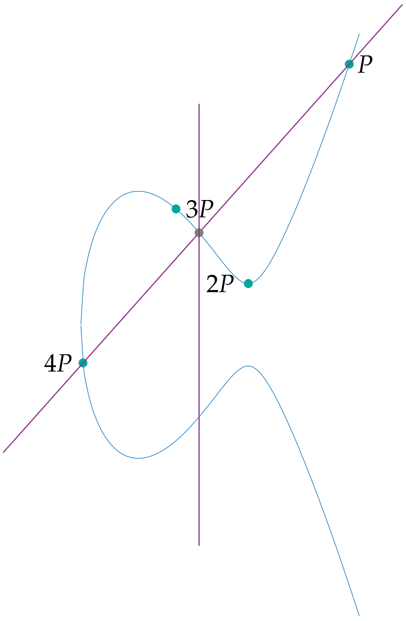

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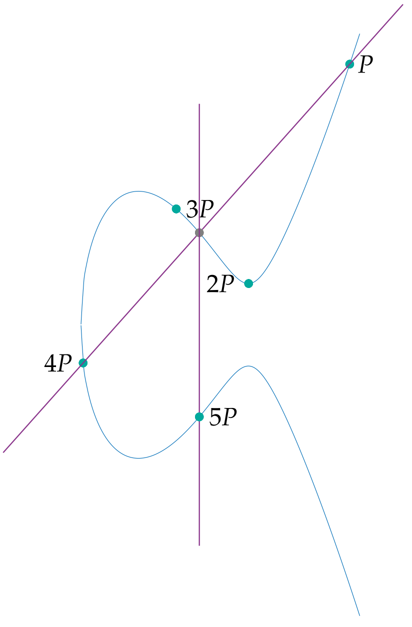

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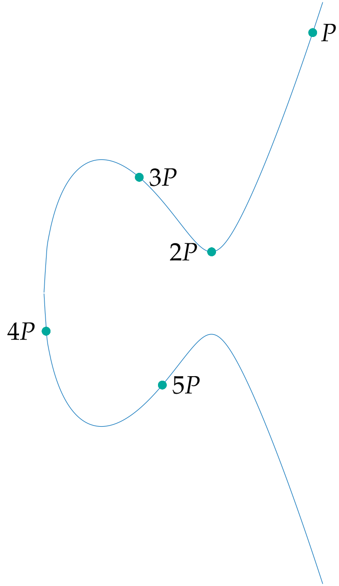

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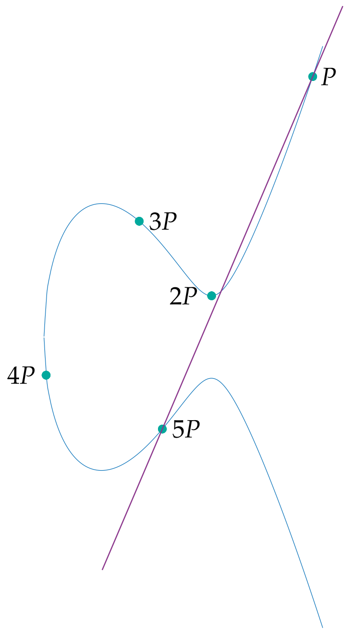

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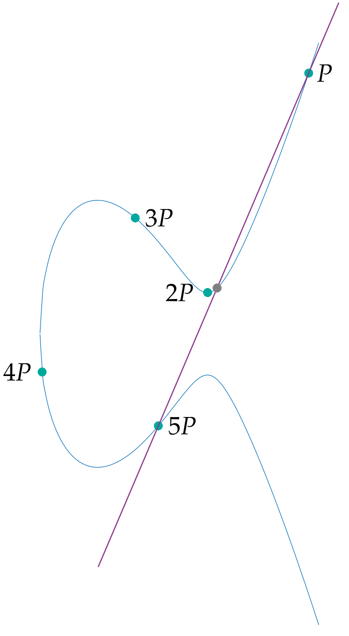

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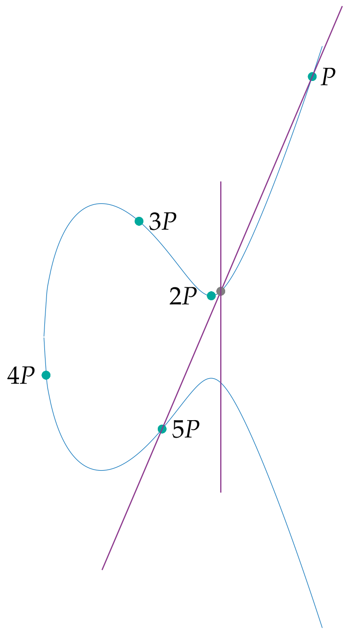

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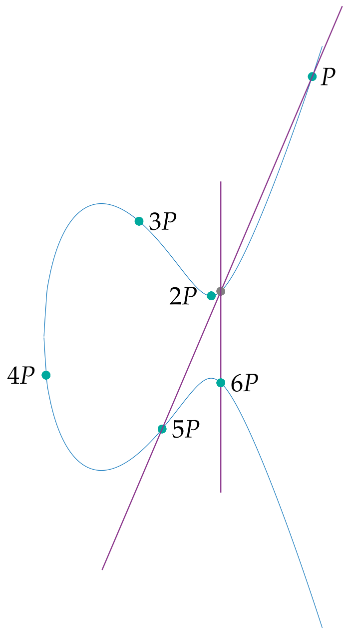

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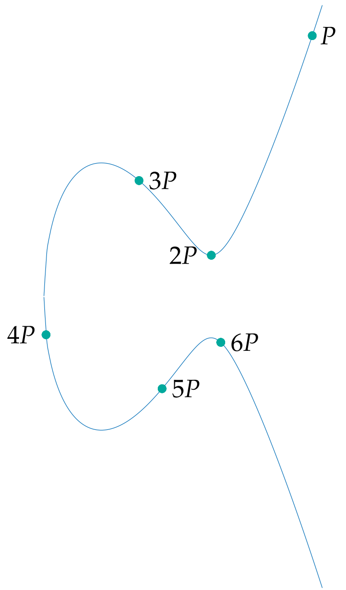

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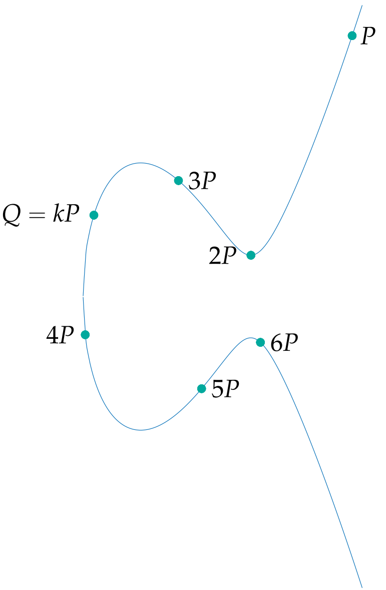
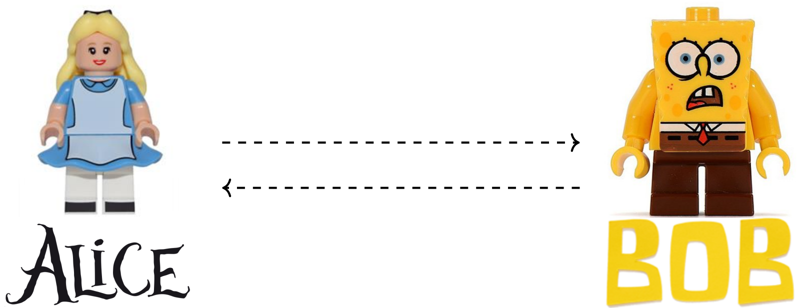
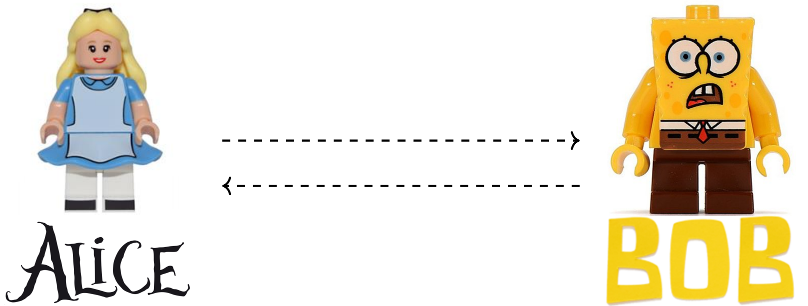


Elliptic Curve Diffie–Hellman



Elliptic Curve Diffie–Hellman

Public information: point $P \in E$



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$Q_{\text{Alice}} = aP$

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ALICE

Q_{Alice} ⇢

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BOB

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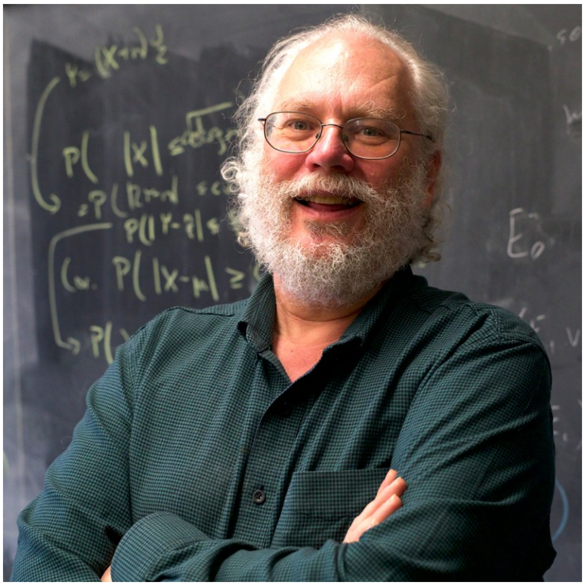
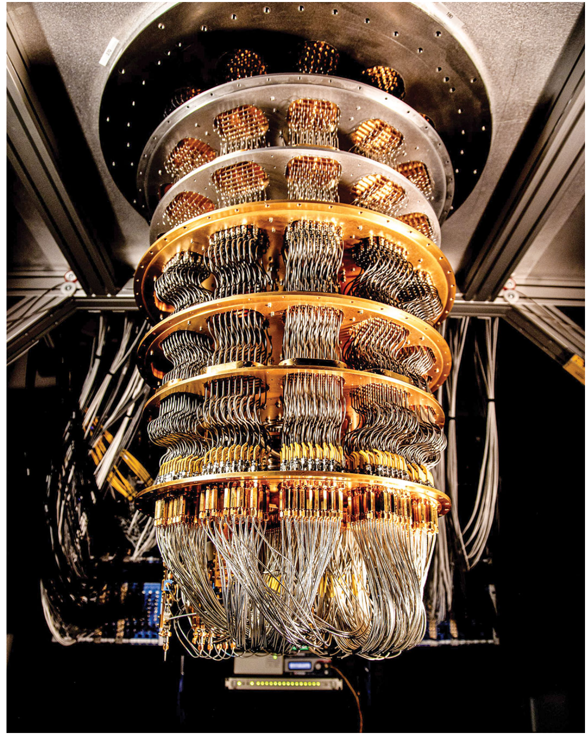
EVE

Shor's quantum algorithm





National Institute of Standards and Technology

NIST initiated a Post-Quantum Cryptography Standardization:

- December 20th, 2016: call to replace ECDH/RSA/…based on new hard problems:
  - finding short vectors in lattices
  - decoding for random linear codes
  - solving nonlinear systems of equations
  - finding isogenies between elliptic curves
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    - ...
- December 21st, 2017: 69 proposals accepted for round 1.
- January 30th, 2019: 26 remainders to round 2.
- July 22nd, 2020: 15 remainders to round 3.

National Institute of Standards and Technology

- July 5th, 2022:
  - 3 winners for digital signatures: CRYSTALS-Dilithium, FALCON, SPHINCS+
  - 1 winner for public key exchange: CRYSTALS-Kyber
  - 4 alternatives for public key exchange to round 4: BIKE, Classical McEliece, HQC, SIKE

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New call for additional signature proposals in September 2022 to promote diversification!

- June 1st, 2023: 40 proposals accepted for round 1

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New call for additional signature proposals in September 2022 to promote diversification!

- June 1st, 2023: 40 proposals accepted for round 1
- October 24th, 2024: 14 remainders for round 2, including SQISign!

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SQISign still remains, the only isogeny-based submission!

- The good:
  - extremely compact (similar to current ECDSA)
  - fast verification
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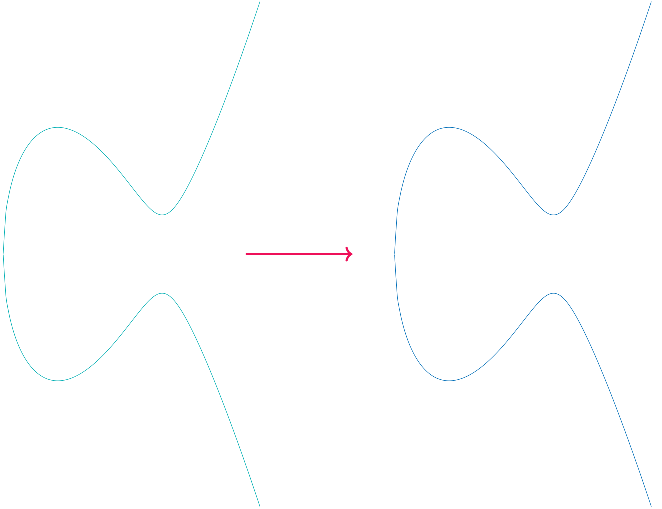
- The ugly:
  - security assumption is complex and rather ad hoc

Isogenies



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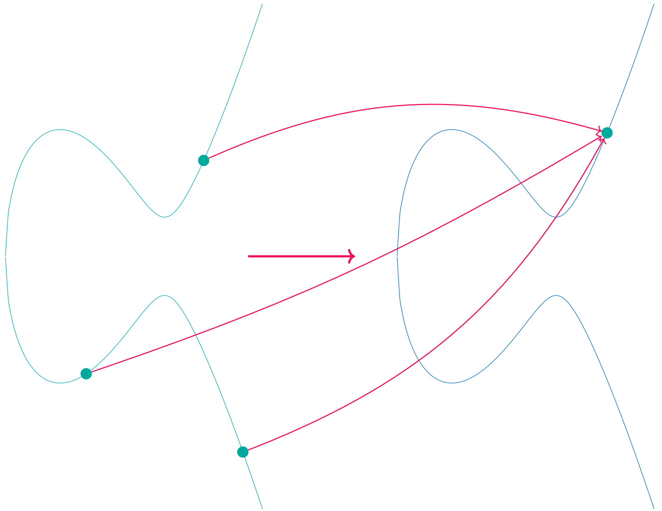

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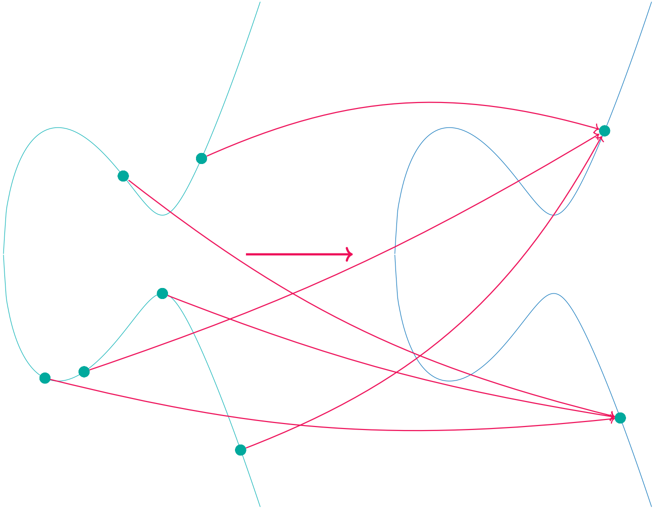

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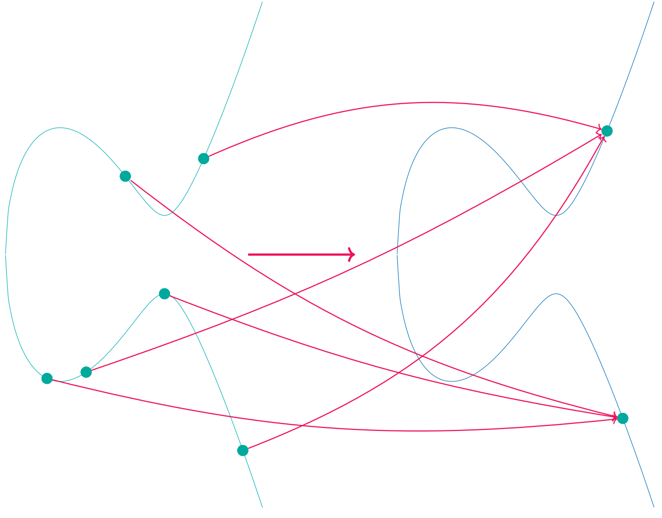
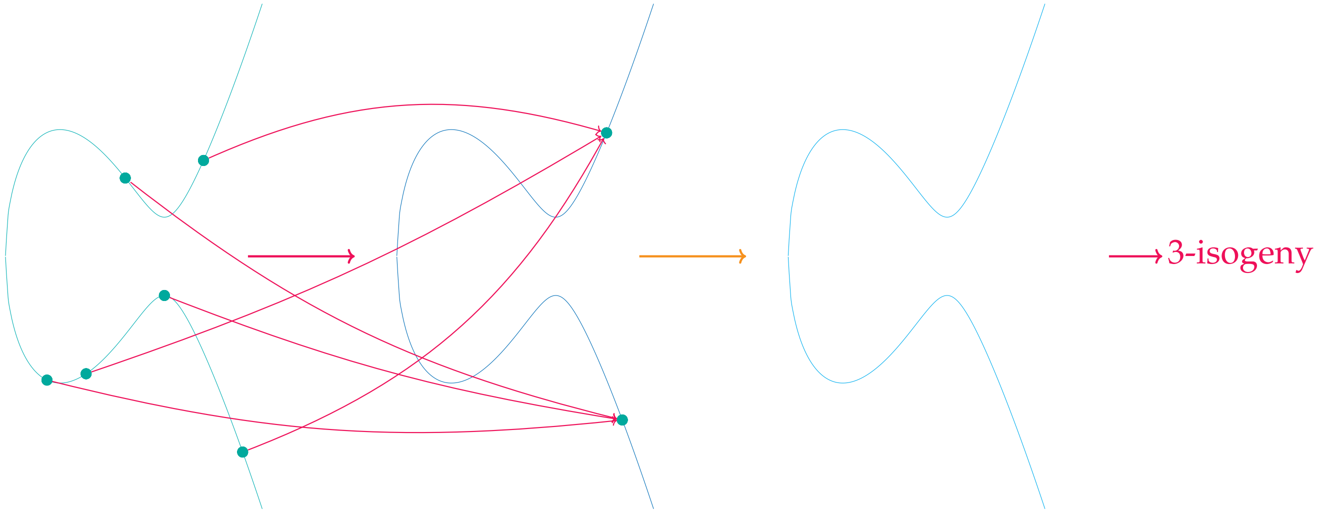


⟶ 3-isogeny

Isogenies



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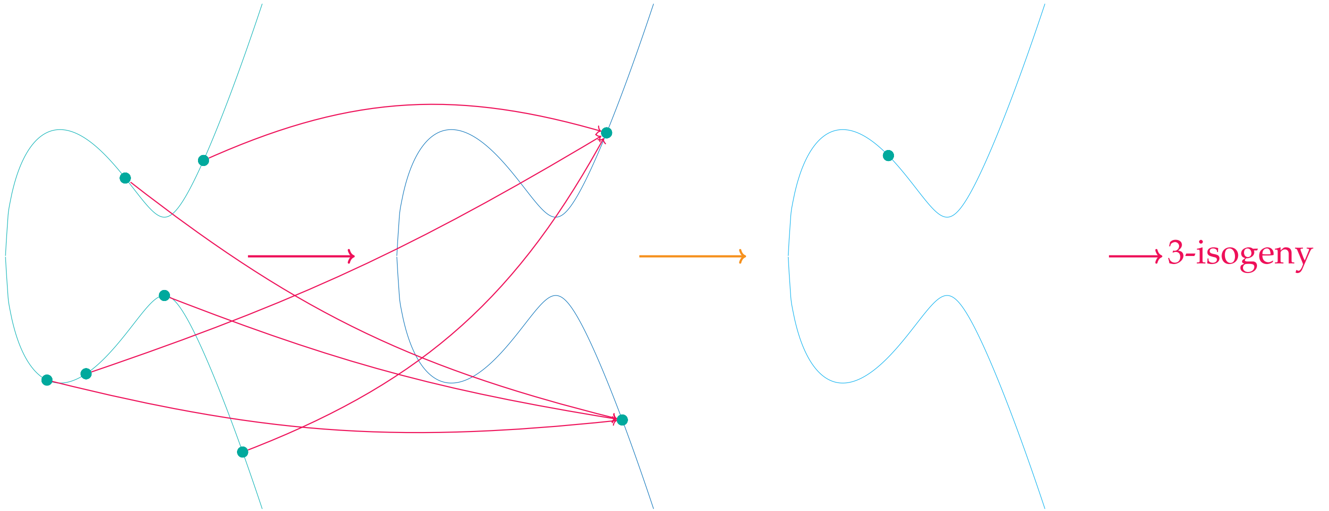

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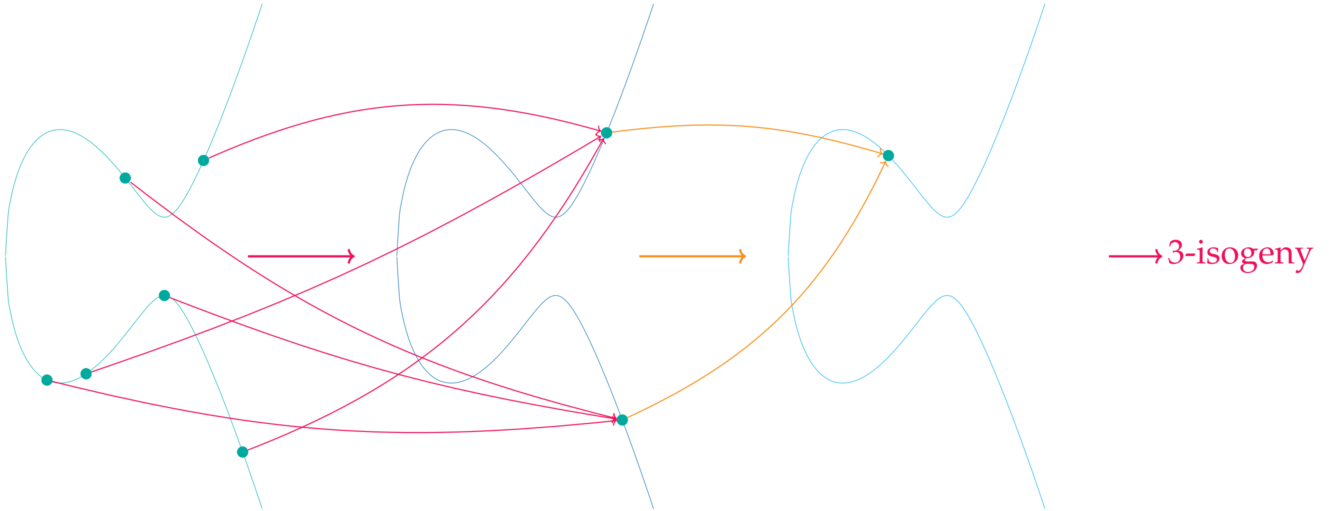

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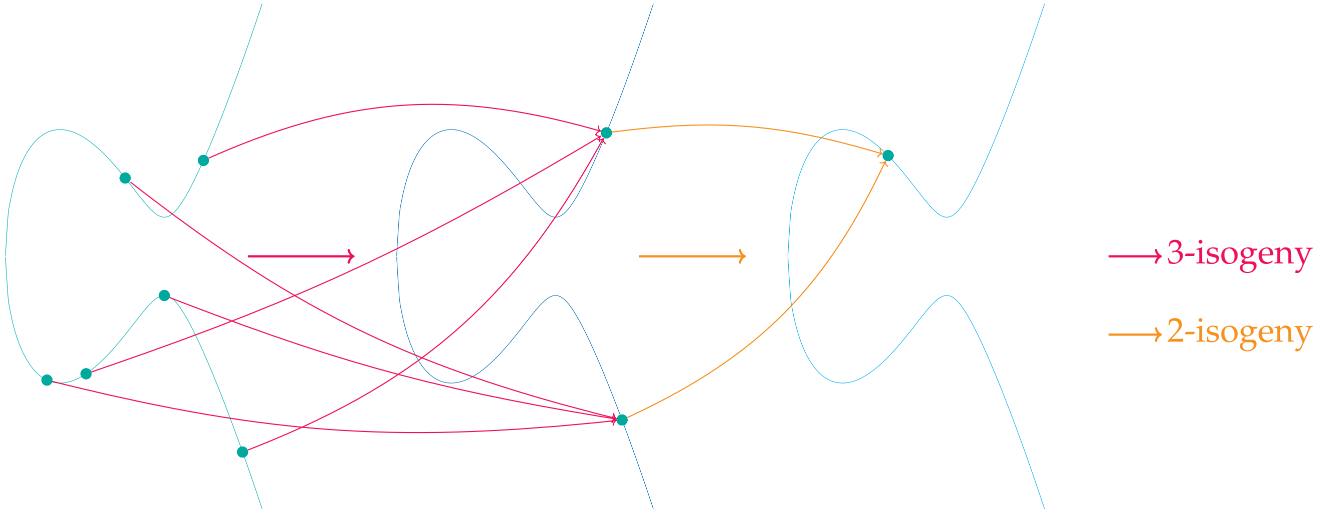

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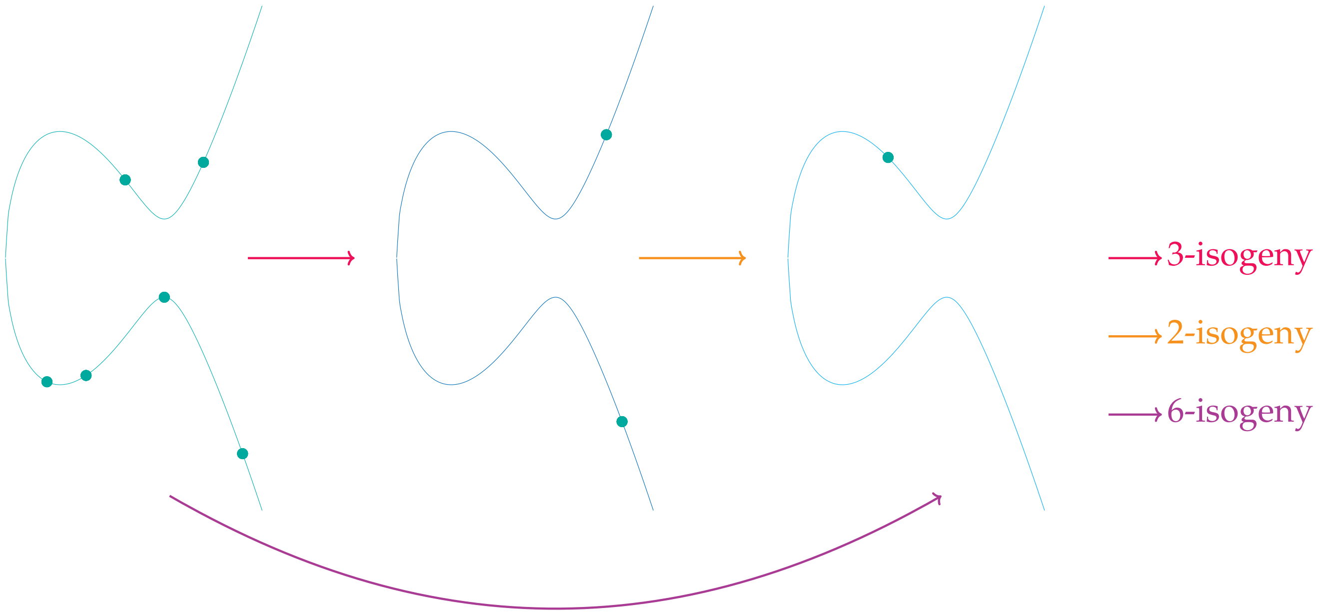

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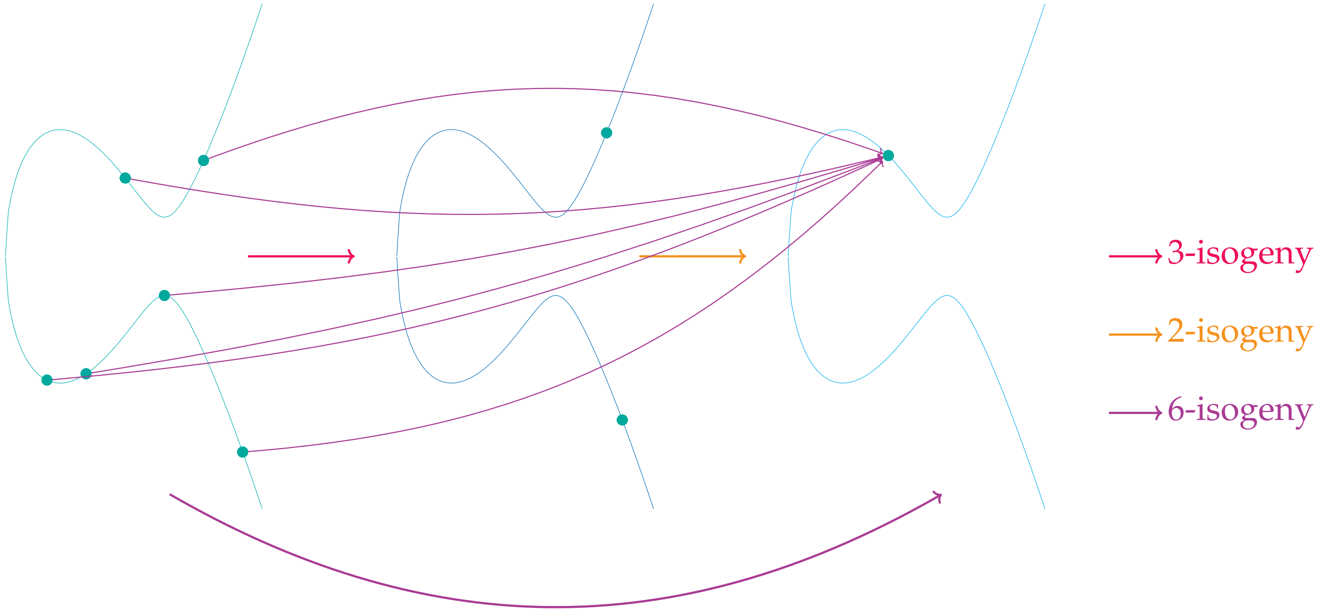

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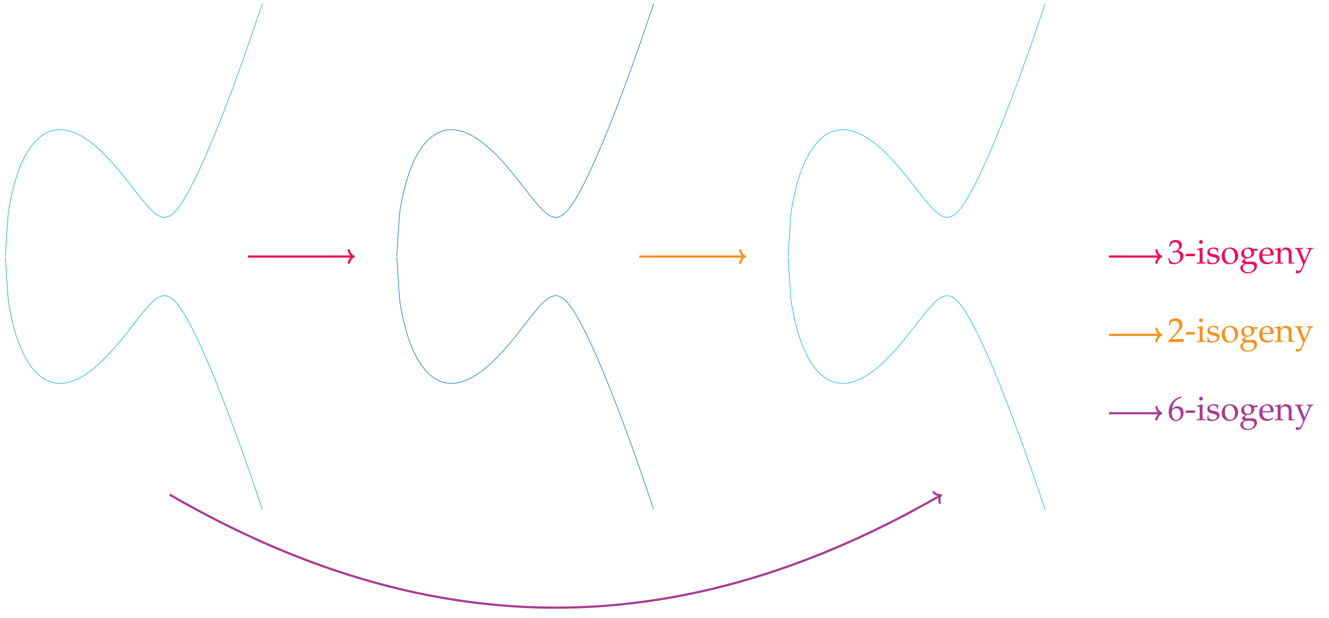

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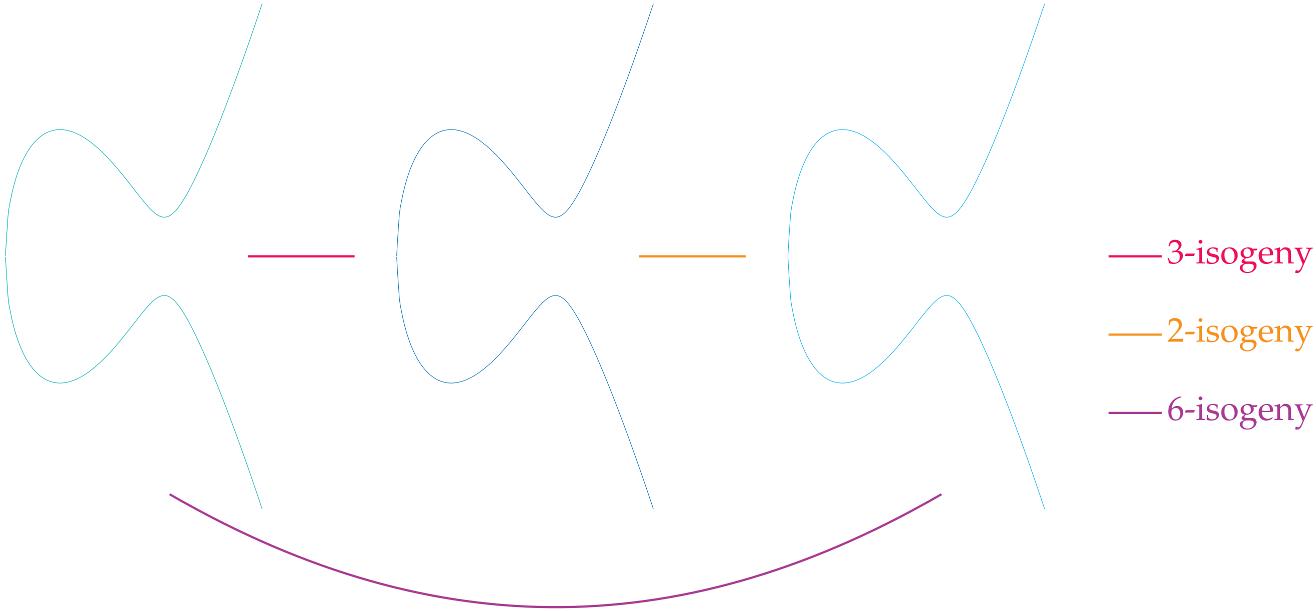

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Hard problems

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Isogenies need to be both represented and evaluated!

- representation typically by $\ker\varphi$ (i.e. all points mapped to neutral element ∞)
- evaluation typically by Vélu-type formulae (i.e. complexity $\mathcal{O}(\deg\varphi)$ or best case $\tilde{\mathcal{O}}(\sqrt{\deg\varphi})$)

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One-endomorphism-finding problem:

Given E supersingular, find one (nontrivial) endomorphism $\varphi : E \to E$.

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Frobenius map:

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Under the Deuring correspondence, there is an isomorphism between the endomorphism ring of supersingular elliptic curves and maximal orders in the quaternion algebra $B_{p,\infty}$, i.e. $\mathbb{Q}(1, i, j, k)$ with

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For the endomorphism ring of E_0, one possible identification is given by

$$[1] \mapsto 1, \qquad \iota \mapsto i, \qquad \pi \mapsto j$$

and then

$$\mathrm{End}(E_0) \cong \mathcal{O}_0 = \left\langle 1, i, \frac{i+j}{2}, \frac{1+k}{2} \right\rangle.$$

Deuring correspondence

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- the endomorphism ring $\mathrm{End}(E_0)$ of a supersingular elliptic curve E_0 is equivalent to a maximal order $\mathcal{O}_0$ in the quaternion algebra $B_{p,\infty}$
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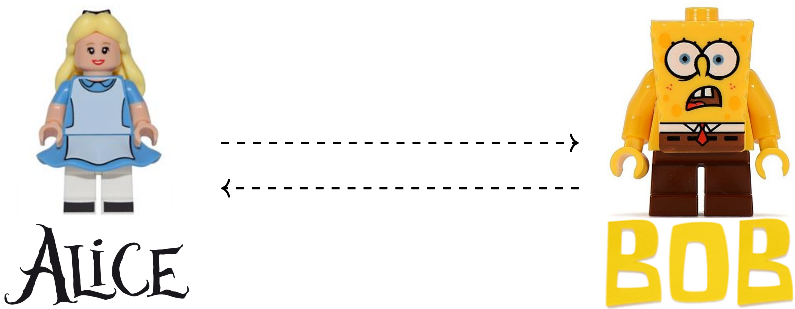
KLPT is an algorithmic tool which allows us to find equivalent ideals $J \sim I$ of different norm!

- The output is a lot larger than optimal, i.e. $\tilde{\mathcal{O}}(p^{3+\varepsilon})$

Isogeny-based Diffie–Hellman?



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CSIDH

We can "make this commutative" by restricting to:

- supersingular elliptic curves defined over $\mathbb{F}_p$ instead of $\mathbb{F}_{p^2}$
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Theorem 1

The class group $cl(\mathcal{O})$ acts freely and transitively on the set of elliptic curves E with $End_{\mathbb{F}_p}(E) \cong \mathcal{O}$, where $\pi \in \mathcal{O}$ corresponds to $\mathbb{F}_p$-Frobenius.

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This results in CSIDH:

- Alice samples $[\mathfrak{a}] \in \mathrm{cl}(\mathcal{O})$ and act on E to get to $[\mathfrak{a}]E$
- Bob samples $[\mathfrak{b}] \in \mathrm{cl}(\mathcal{O})$ and act on E to get to $[\mathfrak{b}]E$
- they both end up on $[\mathfrak{a}\mathfrak{b}]E = [\mathfrak{b}\mathfrak{a}]E$

CSIDH SETTING

The good:

- extremely flexible due to abstraction as group action!

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CSIDH SETTING

The good:

- extremely flexible due to abstraction as group action!

The bad:

- this is essentially the abelian hidden-shift problem so subexponential quantum attacks exist
  - (there's also some controversy about how high parameters should be for this)
- despite speedups and the fact that everything happens over $\mathbb{F}_p$, it's quite slow:
  - you can't randomly sample from $\mathrm{cl}(\mathcal{O})$, so we resort to ideals of the form

$$(3, \pi \pm 1)^{e_1}(5, \pi \pm 1)^{e_2} \dots (587, \pi \pm 1)^{e_{74}}$$

with $e_i \in [-5; 5]$, corresponding to an isogeny of degree (at most)

$$(3 \cdot 5 \cdot \dots \cdot 587)^5$$

SIDH commutative diagram

- Alice and Bob choose (public) bases $\langle P_A, Q_A\rangle = E[2^a]$ and $\langle P_B, Q_B\rangle = E[3^b]$

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$$\begin{array}{ccc} E & \xrightarrow{\varphi_a} & E_{\text{Alice}} \\ \downarrow{\scriptstyle \varphi_b} & & \downarrow{\scriptstyle \theta_b} \\ E_{\text{Bob}} & \xrightarrow{\theta_a} & E_{\text{Alice+Bob}} \end{array}$$

with

- $\ker \theta_a = \langle \varphi_b(P_A) + s_a \varphi_b(Q_A) \rangle$
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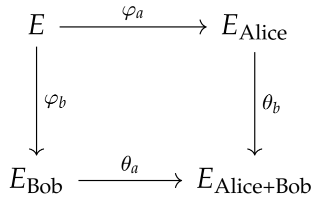

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- $\ker \theta_b = \langle \varphi_a(P_B) + s_b \varphi_a(Q_B) \rangle$
- $\ker(\theta_a \circ \varphi_b) = \ker(\theta_b \circ \varphi_a) = \langle P_A + s_a Q_A, P_B + s_b Q_b \rangle$

Kani's lemma

Lemma.[Ernst Kani, 1997]
Let $\mathbf{f} = (f, H_1, H_2)$ be an isogeny diamond configuration of order N from E_1 to E_2 and put $n = N/d$ and $k_i = n_i/d$, where $d = (n_1, n_2)$ and $n_i = \#H_i$. Then f factors (uniquely) over $[d]$, i.e. $f = \bar{f} \circ [d]$, and there is a unique reducible anti-isometry $\psi = \psi_{\mathbf{f}} : E_1[N] \to E_2[N]$ such that

$$\psi(k_1 x_1 + k_2 x_2) = \bar{f}(x_2 - x_1), \quad \forall x_i \in \widetilde{H}_i = [n]^{-1}(H_i),$$

and every reducible anti-isometry is of this form. Furthermore, if $\mathbf{f}' = (f', H_1', H_2')$ is another isogeny diamond configuration, then we have $\psi_{\mathbf{f}} = \psi_{\mathbf{f}'} \iff \mathbf{f} \sim \mathbf{f}'$.

Kani's lemma

Consider the commutative diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\beta} & E_3 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\gamma} \\ E_2 & \xrightarrow{\delta} & E_4 \end{array}$$

with $\deg \alpha = \deg \gamma$ and $\deg \beta = \deg \delta$

KANI'S LEMMA

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with $\deg \alpha = \deg \gamma$ and $\deg \beta = \deg \delta$, then

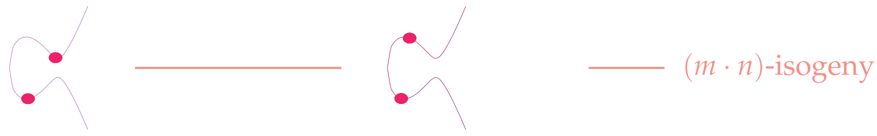
$$\Phi : E_2 \times E_3 \to E_1 \times E_4$$
$$(P, Q) \mapsto \begin{pmatrix} \hat{\alpha} & \hat{\beta} \\ -\delta & \gamma \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix}$$

is a $(\deg \alpha + \deg \beta, \deg \alpha + \deg \beta)$-isogeny between principally polarised abelian surfaces with

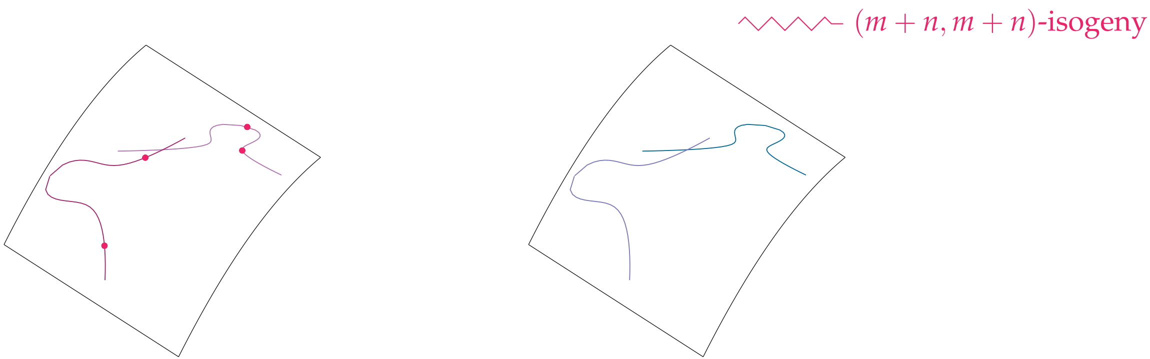
$$\ker \Phi = \{(\alpha(P), \beta(P)) \mid P \in E_1[\deg \alpha + \deg \beta]\}.$$

KANI'S LEMMA APPLIED

Given the one-dimensional isogeny



this determines the two-dimensional isogeny



Attack on SIDH/SIKE

Assume that

- Alice computes a 2^a-isogeny
- Bob computes a 3^b-isogeny $\varphi_B : E \to E_B$ and shares $\varphi_B(P_A), \varphi_B(Q_A)$ as well
- $2^a - 3^b = c^2$ is a perfect square (for simplification purposes)

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$$\begin{array}{ccc} E & \xrightarrow{\varphi_B} & E_B \\ \downarrow{\scriptstyle [c]} & & \downarrow{\scriptstyle [c]} \\ E & \xrightarrow{\varphi_B} & E_B \end{array}$$

where $3^b + c^2 = 2^a$, giving rise to the $(2^a, 2^a)$-isogeny with (known!) kernel

$$\ker \Phi = \{(cP, \varphi_B(P) \mid P \in E[2^a]\}$$

Attack on SIDH/SIKE

To complete the attack:

- compute the $(2^a, 2^a)$-isogeny
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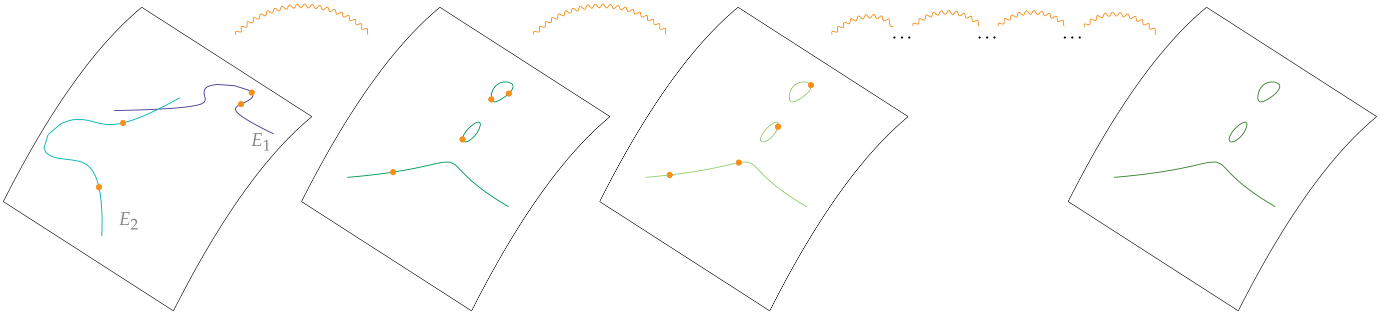
- more generally, you can consider an 8-dimensional isogeny

$$E^4 \times E_B^4 \to E^4 \times E_B^4$$

and take the easy isogenies $[c_1], [c_2], [c_3], [c_4]$ since those exist such that

$$3^b - 2^a = c_1^2 + c_2^2 + c_3^2 + c_4^2$$

Different types of abelian surfaces



Isogeny representations

Several ways to represent degree-d isogeny φ:

- as a rational map $f(x)$, where

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- as $\ker \varphi$, typically through generators, but computations must be feasible
  - e.g. Vélu for large prime d cannot be done
- as kernel ideal I via Deuring correspondence but
  - must be smoothened via KLPT to be useful
  - requires knowledge of endomorphism ring

New isogeny representation

Theorem 2

Let $\varphi : E_1 \to E_2$ *be an isogeny of (known) degree* d*, with interpolation data*

$$P_1, \varphi(P_1), \ldots, P_r, \varphi(P_r)$$

such that $\langle P_1, \ldots P_r \rangle$ *has (smooth) order* $N > 4d$*. Then there exists a polynomial-time algorithm for evaluating* φ*.*

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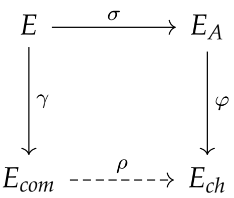
Biggest issue is that polynomial-time is "theoretical":

- sometimes we need to use isogenies in dimension 4 and 8, with the dimension being an exponent in the complexity
- ideally we have parameters such that dimension is 2 and $N = 2^a$

SQISign

Consider the commitment scheme



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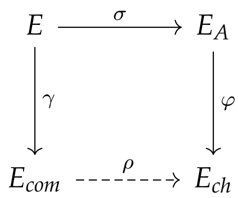

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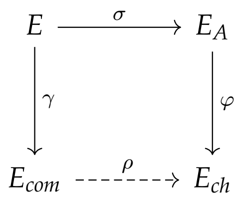

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What works:

- smooth this ρ with KLPT to a different-degree isogeny
- doesn't scale well and zero-knowledge assumption is ad hoc

SQISign higher dimensions

SQISignHD:

- take $\rho : E_{com} \to E_{ch}$ represented by interpolation data for (random) small-degree isogeny

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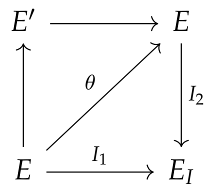

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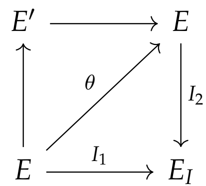

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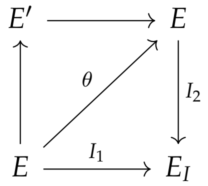

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  allowing us to compute the isogeny from I without smoothening!
- SQISign2D-East, SQISign2D-West, SQIPrime2D with verification in dimension 2!

Current state

One-dimensional isogeny-based cryptography is rather well understood, apart from perhaps

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Higher-dimensional isogenies have given us tools to make new protocols:

- FESTA, QFESTA
- SCALLOP-HD
- SQISignHD, SQISign2D-East, SQISign2D-West, SQIPrime2D
- PRISM
- ...

All of these protocols use a mixture between one-dimensional and higher-dimensional...

Future paths: computations?

Computational cost:

- efficient formulae exist for
  - isogenies of degree 2 and 3 in dimension 2
  - isogenies of degree 2 in dimension 4
- workable formulae exist for
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Would be nice to have:

- more efficient formulae for other degrees/dimensions
  - given how $\tilde{\mathcal{O}}(\sqrt{\deg \varphi})$ in dimension 1, can we expect $\tilde{\mathcal{O}}((\deg \varphi)^{g/2})$ in dimension g?
- constant time for protocols that need it

Future paths: protocols and algorithms in HD?

General question:

- Is it worth it to consider cryptographic protocols strictly in dimension $g > 1$?

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For this we will need new and efficient algorithms:

- faster isogenies in higher dimensions
- algorithmic tools similar to dimension 1:
  - KLPT^2 exists now!

Future paths: protocols and algorithms in HD?

KLPT^2 uses the Ibukiyama–Katsura–Oort correspondence:

- fix a supersingular E_0 with endomorphism ring $\mathcal{O}_0$, then the superspecial principally polarised abelian surfaces (up to polarised isomorphism) are 1–1 with the set

$$\mathrm{Mat}(E_0 \times E_0) := \left\{ \begin{pmatrix} s & r \\ \bar{r} & t \end{pmatrix}, \quad s, t \in \mathbb{Z}_{>0}, r \in \mathcal{O}_0, st - r\bar{r} = 1 \right\} \quad \subset \mathrm{GL}_2(\mathcal{O}_0),$$

  up to the following equivalence relation:

$$g_1 \sim g_2 \in \mathrm{Mat}(E_0 \times E_0) \quad \Leftrightarrow \quad \exists u \in \mathrm{GL}_2(\mathcal{O}_0), \quad u^* g_1 u = g_2$$

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Theorem 3 (KLPT^2)

There exists a polynomial-time algorithm which upon input $g_1, g_2 \in \mathrm{Mat}(E_0 \times E_0)$ and a prime number $\ell \neq p$, under plausible heuristic assumptions, returns $\gamma \in \mathrm{M}_2(\mathcal{O}_0)$ such that

$$\gamma^* g_2 \gamma = \ell^e g_1$$

where $\ell^e \in O(p^{25+\varepsilon})$.

Future paths: graph theory?

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When going to dimension $g > 1$ we definitely want

- supersingular $\rightarrow$ superspecial
- elliptic curves $\rightarrow$ principally polarised abelian varieties

Future paths: graph theory?

In dimension $g > 1$ there are issues if we generalize geometrically/“naively”:

- lots of small cycles making it awkward to walk around “randomly” in the graph
  - two isogenies with kernel $(\mathbb{Z}/(\ell\mathbb{Z}))^2$ can concatenate to one with kernel
    $$\mathbb{Z}/(\ell^2\mathbb{Z}) \times (\mathbb{Z}/(\ell\mathbb{Z}))^2$$
    instead of $(\mathbb{Z}/(\ell^2\mathbb{Z}))^2$

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$$\mathbb{Z}/(\ell^2\mathbb{Z}) \times (\mathbb{Z}/(\ell\mathbb{Z}))^2$$
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FUTURE PATHS: GRAPH THEORY?

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On the bright side, we do have $\mathcal{O}(p^{2g-1})$ vertices:

- in dimension 1 we have $p/12 + \varepsilon$
- in dimension 2 we have $p^3/2880 + \mathcal{O}(p^2)$
- ...

Future paths: graph theory?

Alternative construction for graph:

- let L be a totally real field with strict class number one, e.g. $L = \mathbb{Q}(\sqrt{5})$, and ring of integers $\mathcal{O}_L$, e.g. $\mathcal{O}_L = \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$
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- fix a supersingular E_0 with endomorphism ring $\mathcal{O}_0$
- consider the superspecial principally polarised abelian varieties with real multiplication, i.e.

$$(E^g, \iota : \mathcal{O}_L \to \operatorname{End}(E^g)),$$

  which are the vertices of our graph, with "starting vertex"

$$E \otimes_{\mathbb{Z}} \mathcal{O}_L$$

  and g is the degree of L
- the edges of our graph are given by right ideals I_i of $\mathcal{O}_0 \otimes \mathcal{O}_L$ and we can "walk" in our graph by computing

$$I_i \otimes_{\mathcal{O}_0 \otimes \mathcal{O}_L} (E \otimes_{\mathbb{Z}} \mathcal{O}_L)$$

Future paths: graph theory?

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- connected
- Ramanujan (so optimal rapid mixing)
- k-regular
- you can make it undirected and avoid loops
- avoid the small cycles from the geometric construction

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The "downside" is that we have less vertices, namely

$$\approx 2\left(\frac{p}{4\pi^2}\right)^g d_L^{3/2}.$$

instead of $\mathcal{O}(p^{2g-1})$.

Isogenies: A brand new day

Despite the fall of SIDH/SIKE, things actually improved for the better!

- existing constructions got faster
- cleaner security assumptions
- new toolboxes for protocol constructions
- somewhat uncharted terrain with lots left to discover:
  - more protocols and optimized versions of the current ones
  - computational speedups
  - algebraic and graph-theoretical results

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Isogeny-based cryptography is alive and well with more activity than ever!